Introducing the Fourier transforms

$$\left.\begin{aligned}\phi_u^*(\alpha, z) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \phi_u(x, z)e^{-i\alpha x}\,dx\\ F^*(\alpha) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \frac{dw}{dx}e^{-i\alpha x}\,dx\\ p^*(\alpha) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} p_u(x)e^{-i\alpha x}\,dx\end{aligned}\right\} \tag{7}$$

we find that Eq. (4) is equivalent to the ordinary differential equation $d^2\phi_u^*/dz^2 - (1-M^2)\alpha^2\phi_u^* = 0$ and its solution satisfying Eqs. (5) and (6) is

$$\phi_u^*(\alpha, z) = -\frac{U}{\beta}\frac{F^*(\alpha)e^{-|\alpha|\beta z}}{|\alpha|} \tag{8}$$

where

$$\beta = (1-M^2)^{1/2}$$

Substituting Eq. (8) in the transformed Eq. (3) we have

$$p_u^*(\alpha) = i\frac{\rho U^2}{\beta l}\frac{\alpha F^*(\alpha)}{|\alpha|}e^{-|\alpha|\beta z} \quad \text{at} \quad z = 0$$

and on inversion

$$p_u(x) = i\frac{\rho U^2}{\beta l}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{dw}{d\xi}H(x-\xi, z=0)\,d\xi \tag{9}$$

where

$$H(x, z) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{\alpha}{|\alpha|}e^{-|\alpha|\beta z + i\alpha x}\,d\alpha =$$

$$i\left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty} e^{-|\alpha|\beta z}\sin\alpha x\,d\alpha = i\left(\frac{2}{\pi}\right)^{1/2}\frac{x}{x^2+\beta^2 z^2}$$

Substituting the last expression in Eq. (9), we obtain

$$p_u(x) = -\frac{\rho U^2}{\beta l}\frac{1}{\pi}\int_0^1\frac{dw}{d\xi}\frac{d\xi}{x-\xi} \tag{10}$$

It is assumed that both sides of the panel are exposed to the flow. Proceeding in a similar way as above, it can be shown that, for the lower half-space ($z \leqq 0$), the perturbation velocity potential is

$$\phi_L^*(\alpha, z) = \frac{U}{\beta}\frac{F^*(\alpha)e^{\beta|\alpha|z}}{|\alpha|} \tag{8a}$$

when by simple transformations the dynamic pressure below the panel $p_L = -p_u$, and the total dynamic pressure acting on the panel $\Delta p = p_u - p_L$ equals

$$\Delta p(x) = 2p_u = -\frac{2\rho U^2}{\pi\beta l}\int_0^1\frac{dw}{d\xi}\frac{d\xi}{x-\xi} \tag{11}$$

The expression (10) is also valid for a thin symmetrical airfoil in noncirculatory flow (in this case however $p_L = +p_u$). It was derived in the wing theory by many authors (using other methods), and the integral in Eq. (10) was thoroughly investigated (e.g. [4,5]). To evaluate this improper integral its principal value should be taken, and its singularities at the leading and trailing edges (which vanish only in the particular case when the panel is clamped along both edges) have a negligible effect. These integrable singularities entail no mathematical difficulties in the subsequent calculations.

Introducing dimensionless coefficients

$$\lambda^2 = \rho U^2 l^3/D(1-M^2)^{1/2}, \qquad s = N_x l^2/\pi^2 D = N_x/N_{EO}$$

where N_{EO} is the critical ("Euler") force for the panel in a fluid at rest, and substituting Eq. (11) in Eq. (1) we have

$$\frac{d^4w}{dx^4} + \pi^2 s\frac{d^2w}{dx^2} - 2\frac{\lambda^2}{\pi}\int_0^1\frac{dw}{d\xi}\frac{d\xi}{x-\xi} = 0 \tag{12}$$

Equation (12) is discussed in Ref. 6 where the aerodynamic operator Eq. (10) was adopted without derivation, from the theory of thin airfoils. Following Ref. 6 we assume that

$$w(x) = \sum_{n=1}^{N} a_n \sin n\pi x \tag{13}$$

thus satisfying the boundary conditions, Eqs. (2). Substituting Eq. (13) in Eq. (12) and using Galerkin's method, a system of N algebraic equations is obtained in the N unknown coefficients a_n. The system has a nontrivial solution provided its determinant vanishes, thus yielding the characteristic equation

$$\text{Det}\,\|\pi^4(k^4 - sk^2)\delta_{nk} - 4\lambda^2 A_{nk}\| = 0 \qquad n, k = 1, 2, \ldots, N$$

where δ_{nk} is the Kronecker symbol and

$$A_{nk} = k\int_0^1 \sin n\pi x\int_0^1\frac{\cos k\pi\xi}{x-\xi}\,d\xi\,dx \tag{14}$$

The diagonal terms A_{nn} turned out to be much larger than the nondiagonal ones, and calculations showed that the lowest critical parameters are obtainable to a high degree of accuracy using the first mode only:

$$\pi^4(1-s) - 4\lambda^2 A_{11} = 0 \tag{15}$$

where † $A_{11} = 1.2046$.

Equation (15) implies that in the absence of compressive forces ($s \equiv 0$), divergence of the panel occurs at the critical flow velocity U_{DO} defined by

$$\lambda_{DO}^2 = \pi^4/4A_{11} = 20.23 \tag{16}$$

At lower flow velocities Eq. (15) can be rewritten in the following symmetrical form

$$N_{cr} = N_{EO}[1 - (\lambda^2/\lambda_{DO}^2)] \qquad \lambda^2 \leqq \lambda_{DO}^2 \tag{17}$$

or explicitly

$$N_{cr} = N_{EO} - (1/2.05)(\rho U^2 l)/(1-M^2)^{1/2} \tag{18}$$

Extreme caution must be exercised in extending the simple formula (17) to other configurations, especially to panels with finite aspect ratios. Systematic numerical calculations are necessary to ascertain how far such extension is permissible.

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† Numerical results obtained (up to $N = 6$) are somewhat different from those listed in Ref. 6.

Square-Root Variable Metric Method for Function Minimization

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Introduction

ONE of the best known methods of minimizing a function of n variables is classified as a variable metric method.[1,2] Probably the best known of these methods is Davidon's method.

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These methods are very efficient parameter optimization schemes. The main drawback of the methods is the extremely accurate one-dimensional search required during each iteration. If the search is not done very accurately, the H matrix associated with the methods may become nonpositive definite and difficulties are encountered by the methods. Several attempts have been made to reduce the accuracy required for the search.[3,4] This Note presents an alternate technique which may be used to eliminate the extreme accuracy required during the search. It involves writing the H matrix as the product of a matrix times itself. If this new matrix is updated, the product remains positive semidefinite regardless of the accuracy obtained during the search. This approach is the same one used in estimation theory to iteratively update the covariance matrix.[6] It is generally referred to as square-root filtering in that case.

Method

The general problem to be solved is the minimization of $G(x)$ where x is an n vector. An initial guess is made for x. The variable metric methods calculate changes in x from the equation

$$\delta x = -\alpha H G_x^T \tag{1}$$

where $\delta x = x^{i+1} - x^i$, α is a scalar, and H is an $n \times n$ symmetric positive definite matrix. The matrix H is updated after each iteration. The scalar α is chosen during each iteration to minimize $G(x^{i+1})$. If the value of α is not determined extremely accurately on every iteration, then H may become nonpositive definite. If this happens then it is not possible to guarantee that G may be decreased on every iteration. At this point, either H must be reinitialized or some other procedure must be used in order to continue iterating toward a solution.

It is possible, however, to rewrite the update equation for H so that it remains positive semidefinite regardless of the accuracy obtained during the search procedure. This can be seen by observing that several of the update equations for H are similar to those for the covariance matrix in estimation theory. It is possible to write the covariance update formula in square-root form. Thus a matrix is updated and then multiplied by itself. This keeps the covariance positive semidefinite. This same procedure may be applied to the update equation for H. The easiest update formula to apply this procedure to is method IV in Ref. 1. Only this update equation will be considered here. The update for H is written as

$$H = H + \frac{(\delta x - H\Delta G_x^T)(\delta x - H\Delta G_x^T)^T}{(\delta x - H\Delta G_x^T)^T \Delta G_x^T} \tag{2}$$

where $\Delta G_x = G_x^{i+1} - G_x^i$. Since $\delta x^i = -\alpha H G_x^{iT}$ this equation may be rewritten as

$$H = H - (1/a) H y y^T H \tag{3}$$

where $a = y^T H \Delta G_x^T$ and $y = \alpha G_x^{iT} + \Delta G_x^T$. It is now possible to write $H = SS^T$ where S is lower triangular. Note that if an update formula for S is found, then H will remain symmetric and positive semidefinite. If Eq. (3) is written in terms of S, then

$$SS^T = S[I - (1/a) S^T y y^T S] S^T \tag{4}$$

The term in brackets is then written as

$$[I - (1/a) S^T y y^T S] = [I - (v/a) S^T y y^T S][I - (v/a) S^T y y^T S]^T \tag{5}$$

where v is determined to ensure the expression on the right is equivalent to the one on the left. The solution for v requires

$$v = (a/b)\{1 \pm [1 - (b/a)]^{1/2}\} \tag{6}$$

The plus sign in Eq. (6) is chosen for convenience. Thus

$$S = S[I - (v/a) S^T y y^T S] \tag{7}$$

and $H = SS^T$. Note that v gives an indication of whether or not H would have been positive semidefinite if Eq. (2) had been used. If v is real, then S is real and H is positive semidefinite. This is the desired property. If v becomes imaginary, H will not be real and positive semidefinite. If $(1 - b/a) < 0$ then choose $v = a/b$. Note that v becomes imaginary as the quantity under the square root changes sign. Hence setting it to zero is a reasonable choice in this case. This keeps H positive semidefinite. Note that this choice will destroy the quadratic convergence characteristic of the method. When $v = a/b$ the method behaves as a weighted gradient scheme. However, in the vicinity of a minimum, values of v are expected to be real. Hence good convergence characteristics are expected when near a minimum.

Example

The method described above was used to solve most of the test functions described in Ref. 4. The results of this study are shown by Wang.[8] A parabolic search technique was used with the method. This consisted of increasing or decreasing an initial guess for α by a factor of 10 until a minimum value was bracketed. Then parabolic interpolation was used to find the minimum value. The search was terminated when normalized changes in either α or G were below a prescribed tolerance, N. The results were compared with a Davidon program described in Ref. 5. This program uses a sophisticated one-dimensional search consisting of Golden Section and cubic interpolation. Most of the examples in Ref. 4 were solved using both techniques. A value of $N = 0.1$ was used for the SRVM (Square-Root Variable Metric) method and $N = 1.0\text{E-}11$ for the Davidon method. In general, the SRVM method required more derivative evaluations but fewer G evaluations during the search than the Davidon method. The total number of function evaluations and computer time used were about the same for the methods.

An additional example considered was the lunar ascent problem described in Ref. 7. The problem involves finding the steering angle to minimize the time to place a constant thrust magnitude vehicle into orbit about the moon. If a linear steering law is assumed, the differential equations may be integrated and the problem becomes a parameter optimization problem. This requires that I be minimized where

$$I = t_f + (w_1/2)[-\tfrac{1}{2} g t_f^2 - T[\sin(a_o t_f + a_1) - \sin(a_1)]/a_o^2 + T t_f (\cos a_1)/a_o - Y_f]^2 + (w_2/2)[T t_f (\cos a_1)/a_o - \dot{X}_f]^2$$

Here T is the thrust to mass ratio, g is the lunar gravitational constant, Y_f is the fixed final altitude, $\dot{X}_f$ is the fixed final horizontal velocity component, and w_1 and w_2 are weights. The parameters are t_f, a_o, and a_1. The penalty function approach is used for the constraint equations involving fixed values for Y_f and $\dot{X}_f$. This problem appears to be troublesome due to the fairly large variation in parameters. Thus $I_{t_f t_f}$ is on the order of 10^{-13} and $I_{a_1 a_1}$ is approximately 10^{-3}. The Davidon scheme does not solve this problem. The SRVM method solves the problem in 13 iterations.

Conclusions

The new method can easily be implemented in current Davidon programs. If the number of variables is not sufficiently large so that accuracy is lost in forming SS^T this approach always generates a positive semidefinite H matrix. Thus the descent property is maintained regardless of the accuracy obtained in the one-dimensional search. If an accurate search technique is used, then the SRVM method is equivalent to the Davidon scheme as long as H remains positive definite. If the Davidon scheme does not keep H positive definite, however, the SRVM method will. The SRVM method required approximately the same amount of time as the Davidon scheme for most test functions. It also solved the lunar ascent problem which the Davidon scheme did not solve.

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Bolotin's Method Applied to the Buckling and Lateral Vibration of Stressed Plates

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Introduction

In a recent Note by King and Lin[1] the edge effect method proposed by Bolotin[2] was applied to the problem of the lateral vibration of rectangular orthotropic and isotropic plates subject to various boundary conditions. The method was shown to be remarkably accurate when used for determining the natural frequencies of such plates. A natural extension of this work is to the buckling and lateral vibration of rectangular plates subject to in-plane forces which are time invariant and constant over the area of the plate and have their principal directions parallel to the plate edges. It is the purpose of this Note to illustrate this extension and to demonstrate its applicability by the presentation of numerical results for a particular plate.

Development of the Analysis

The development of the analysis follows the steps outlined by King and Lin[1] or, alternatively, as presented by Dickinson and Warburton[3] in their paper on the edge effect method applied to plate systems. The plate is assumed to lie in the xy-plane, to be bounded by edges $x = 0, a$ and $y = 0, b$; to be of a uniform thickness, rectangularly orthotropic material having its symmetry axes orthogonal to the plate boundaries and acted upon by constant in-plane forces per unit width N_x and N_y (tensile positive) acting in the x and y directions, respectively.

For free vibration, the lateral displacement $w(x,y)e^{i\omega t}$ is governed by the equation

$$D_x \partial^4 w/\partial x^4 + 2H\partial^4 w/\partial x^2 \partial y^2 + D_y \partial^4 w/\partial y^4 - N_x \partial^2 w/\partial x^2 - N_y \partial^2 w/\partial y^2 - \rho\omega^2 w = 0 \qquad (1)$$

where D_x, H, D_y are plate flexural rigidities and ρ is the plate mass/unit area. Considering the effects of the edges to be localized, the displacement away from the edges may be represented by $w = f(x)g(y) = W_0 \sin k_x(x/a-\alpha)\sin k_y(y/b-\beta)$ which satisfies Eq. (1) provided that

$$\omega^2 = (1/\rho)[D_x(k_x/a)^4 + 2H(k_x k_y/ab)^2 + D_y(k_y/b)^4 + N_x(k_x/a)^2 + N_y(k_y/b)^2] \qquad (2)$$

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Near to an edge an exponential term is included in $f(x)$ [or $g(y)$], which decays away from the edge. Thus, in the vicinity of $x = 0$, an additional term $Ae^{\gamma_x x/a}$ is included in $f(x)$ and near $y = 0$ a term $Be^{\gamma_y y/b}$ in $g(y)$. For these additional terms to be admissible by Eq. (1) and for γ to be real and negative, it is necessary that

$$\gamma_x = -[k_x^2 + 2(H/D_x)(k_y a/b)^2 + N_x a^2/D_x]^{1/2}$$

and

$$\gamma_y = -[k_y^2 + 2(H/D_y)(k_x b/a)^2 + N_y b^2/D_y]^{1/2}$$

Near to the edge $x = 0$, $f(x)$ may be written

$$f(x) = A_1 \sin k_x x/a + A_2 \cos k_x x/a + A_3 e^{\gamma_x x/a}$$

and near to edge $x = a$,

$$f(x) = A_4 \sin k_x(1-x/a) + A_5 \cos k_x(1-x/a) + A_6 e^{\gamma_x(1-x/a)}$$

Similar expressions may be written for $g(y)$ near $y = 0$ and $y = b$, involving six constants B_1 to B_6.

Continuity of displacement functions away from the plate edges leads to two equations giving A_4 and A_5 in terms of A_1 and A_2 and two similar equations giving B_4 and B_5 in terms of B_1 and B_2. The 12 arbitrary constants A_s and B_s are thus eliminated by consideration of the eight plate boundary conditions and the four continuity conditions, resulting in two simultaneous equations in k_x and k_y, the solution of which enables ω^2 to be determined from Eq. (2).

In the case of buckling, $\omega = 0$; thus, if either N_x or N_y is known, or if the relationship between N_x and N_y is known, the appropriate buckling values are readily obtainable.

Numerical Results

In order to demonstrate the application of the edge effect method to plates subject to in-plane forces, the method was used to determine the buckling loads and fundamental natural frequencies of a square, clamped plate subject to hydrostatic in-plane force ($N_x = N_y = N$), for various conditions of orthotropy. Corresponding, accurate natural frequencies had previously been determined by the author using a series solution[4] and were thus available for comparison.

The simultaneous equations which result for the clamped plate are

$$2\cos k_x + (\gamma_x/k_x - k_x/\gamma_x)\sin k_x = 0$$

and

$$2\cos k_y + (\gamma_y/k_y - k_y/\gamma_y)\sin k_y = 0$$

the solutions of which were obtained using a numerical interpretation of Taylor's theorem.

Table 1 shows the fundamental frequency parameters computed using the edge effect method and the series solution and quite close agreement may be seen to be achieved. From the nature of the edge effect method, as discussed in Refs. 1 and 3,

Table 1 Fundamental frequency parameter $\omega ab(\rho/H)^{1/2}$ for a square clamped orthotropic plate subject to hydrostatic in-plane force (tension positive)

		D_x/H					
		1/2		1		2	
$Na^2/\pi^2 H$	D_y/H	Edge effect	Series[4]	Edge effect	Series[4]	Edge effect	Series[4]
−2		15.78	17.74				
0	1/2	27.47	28.07				
10		54.93	54.98				
20		71.89	71.91				
−2		22.04	23.78	26.73	28.57		
0	1	31.56	32.27	35.09	35.99		
10		57.42	57.50	59.80	59.93		
20		74.04	74.07	76.13	76.17		
−2		31.20	32.66	34.59	36.30	40.90	42.64
0	2	38.54	39.29	41.42	42.40	46.83	47.96
10		61.80	61.92	64.00	64.18	67.91	68.17
20		77.71	77.75	79.70	79.77	83.10	83.21